

ON THE RANKS OF THE ALGEBRAIC K -THEORY OF HYPERBOLIC GROUPS

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Dedicated to Fico on the occassion of his 70th birthday.

ABSTRACT. Let G be a word hyperbolic group. We prove that the algebraic K -theory groups of $\mathbb{Z}[G]$, $K_n(\mathbb{Z}[G])$, have finite rank for all $n \in \mathbb{Z}$. For a few classes of groups, we give explicit formulas for the ranks of the algebraic K -theory groups of their group rings.

1. INTRODUCTION AND PRELIMINARIES

Recall that the Farrell-Jones isomorphism conjecture proposes that, for any discrete group G , the algebraic K -theory of the group ring $\mathbb{Z}[G]$ is determined by the algebraic K -theory of the virtually cyclic subgroups of G plus homological information.

Conjecture 1. (*Farrell-Jones Isomorphism Conjecture, IC*)

Let G be a discrete group. Then for all $n \in \mathbb{Z}$ the assembly map

$$(1) \quad A_{V_{cyc}} : H_n^G(\underline{EG}; \mathcal{K}) \rightarrow H_n^G(pt; \mathcal{K}) \cong K_n(\mathbb{Z}[G])$$

induced by the projection $\underline{EG} \rightarrow pt$ is an isomorphism, where $H_^G(-; \mathcal{K})$ is a suitable equivariant homology theory with local coefficients in \mathcal{K} , the non-connective spectrum of algebraic K theory and \underline{EG} is a model for the classifying space for actions with isotropy in the family of virtually cyclic subgroups of G .*

This conjecture has been verified, among others, when G is a word hyperbolic group [BLR08], or a $CAT(0)$ -group [Weg12]. Once we know the conjecture holds for a group G , we can try to compute $K_n(\mathbb{Z}[G]) \cong H_n^G(\underline{EG})$ using an Atiyah-Hirzebruch type spectral sequence.

In this paper we use the validity of the Farrell-Jones conjecture and the corresponding spectral sequence to show that the rank of $K_n(\mathbb{Z}[G])$ is finite for all $n \in \mathbb{Z}$, where G is a word hyperbolic group. Next, we give some explicit examples of computations of these ranks.

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For hyperbolic groups Leary and Juan-Pineda [JPL06], showed that

$$(2) \quad H_n^G(\underline{EG}; \mathcal{K}) \cong H_n^G(\underline{EG}; \mathcal{K}) \oplus \bigoplus_{(V)} \text{cok}_n(V),$$

where \underline{EG} is the classifying space for the family FIN , of finite subgroups of G , (V) consists of one representative from each conjugacy class of maximal infinite virtually cyclic subgroup of G and $\text{cok}_n(V)$ is the cokernel of the homomorphism $H_n^V(\underline{EV} \rightarrow pt; \mathcal{K})$.

It is well known, see [Gru08, Thm. 5.9 and page 4.] that the terms $\text{cok}_n V$ are torsion groups. This gives the following:

Lemma 2. *Let G be a discrete word hyperbolic group. Then for all $n \in \mathbb{Z}$*

$$\text{rank}(K_n(\mathbb{Z}[G])) = \text{rank}(H_n^G(\underline{EG}; \mathcal{K})).$$

This paper is, in part, complementary to [JPLP11] where we treated the case lower K groups, namely $K_i()$ for $i \leq 1$. Here we treat the whole spectrum of K theory and a broader class of groups.

2. RANKS.

In view of Lemma 2 the rank of the algebraic K -theory groups of $\mathbb{Z}[G]$ are determined by the ranks of the algebraic K -theory of the finite subgroups of G and the homology of \underline{EG} . The ranks of the algebraic K groups of the group ring of a finite group are given as follows:

Theorem 3. ([Jah09], [Bas65], [Car80a], [Car80b]) *Let H be a finite group with r distinct real irreducible representations, c of them of complex type, and q distinct rational irreducible representations. For $n > 1$ we then have*

$$\text{rank}(K_n(\mathbb{Z}[H])) = \begin{cases} r & \text{if } n \equiv 1 \pmod{4}, \\ c & \text{if } n \equiv 3 \pmod{4}, \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

When $n \leq 1$, we have:

$$\text{rank}(K_1(\mathbb{Z}[H])) = r - q,$$

$$\text{rank}(K_0(\mathbb{Z}[H])) = 1,$$

$$\text{rank}(K_{-1}(\mathbb{Z}[H])) < \infty \text{ and}$$

$$K_{-n}(\mathbb{Z}[H]) = 0 \text{ } n > 1.$$

Note that r is equal to the number of real conjugacy classes of H , that is, classes of the form $C(h) = \{ghg^{-1}, gh^{-1}g^{-1} | g \in H\}$, c is equal to the number of real conjugacy classes such that $C(h) \neq \{hgh^{-1} | h \in H\}$, and q is the number of conjugacy classes of cyclic subgroups of H , see [Ser77].

To compute the equivariant homology groups $H_*^G(\underline{EG}; \mathcal{K})$ we may use an Atiyah-Hirzebruch type spectral sequence. Let C_n denote the set of n -cells of the space $\underline{BG} = \underline{EG}/G$, then the first page of our spectral sequence is given by

$$\begin{array}{ccc}
 \vdots & & \vdots \\
 \cdots \longleftarrow \bigoplus_{\sigma^p \in C_n} K_q(\mathbb{Z}[G_{\sigma^p}]) \longleftarrow \bigoplus_{\sigma^{p+1} \in C_{n+1}} K_q(\mathbb{Z}[G_{\sigma^{p+1}}]) \longleftarrow \cdots \\
 \cdots \longleftarrow \bigoplus_{\sigma^p \in C_n} K_{q-1}(\mathbb{Z}[G_{\sigma^p}]) \longleftarrow \bigoplus_{\sigma^{p+1} \in C_{n+1}} K_{q-1}(\mathbb{Z}[G_{\sigma^{p+1}}]) \longleftarrow \cdots \\
 \vdots & & \vdots
 \end{array}$$

where G_σ denotes the stabilizer of a pre-image $\sigma' \in \underline{EG}$ of $\sigma \in \underline{BG}$, and the homomorphisms in the chain complex are induced by the natural inclusions (up to conjugacy). Note that G_σ is always a finite group, hence we can apply Theorem 3 to every group appearing in our spectral sequence. As a consequence we may identify the second page of the Atiyah-Hirzebruch spectral sequence as

$$E_{p,q}^2 = \mathbb{H}_2(\underline{BG}; \{\mathcal{K}_q(\mathbb{Z}[G_\sigma])\}),$$

where the above is a homology theory with local coefficients given by the algebraic K groups of the group rings $\mathbb{Z}[G_\sigma]$ for all the finite isotropy groups G_σ .

Theorem 4. *Let G be an hyperbolic group. Then $\text{rank}(K_n(\mathbb{Z}[G]))$ is finite for all $n \in \mathbb{Z}$.*

Proof. It is known that for G word hyperbolic, there exists a finite model for \underline{EG} , i.e., such that \underline{BG} is compact, see for example [MS02]. Take this finite model for \underline{EG} . Hence the only possible non-zero terms in the n th page of our spectral sequence $E_{p,q}^n$ are those terms with

$0 \leq p \leq m$, $m = \dim \underline{BG}$, that is, they are contained in a vertical strip for all $n \in \{1, 2, 3, \dots\} \cup \{\infty\}$. Now, since $E_{p,q}^\infty$ has finite rank because it is the quotient of a subgroup of the abelian group $E_{p,q}^1$ and

$$\text{rank}(K_n(\mathbb{Z}[G])) = \sum_{p+q=n} \text{rank}(E_{p,q}^\infty)$$

the proof follows by Theorem 3 and the compactness of \underline{BG} . \square

Note that using [Gru08, Thm. 5.11] Lemma 2 is valid for every group satisfying the Farrell-Jones conjecture, hence following the proof of Theorem 4 we have:

Theorem 5. *Let G be a group that admits a finite model for \underline{BG} and such that satisfies the Farrell-Jones conjecture. Then $\text{rank}(K_n(\mathbb{Z}[G]))$ is finite for all $n \in \mathbb{Z}$.*

This last Theorem is more general and applies, for instance, to the groups that appear in [LO09].

3. EXAMPLES.

In this section we give some explicit computations of $\text{rank}(K_n(\mathbb{Z}[G]))$.

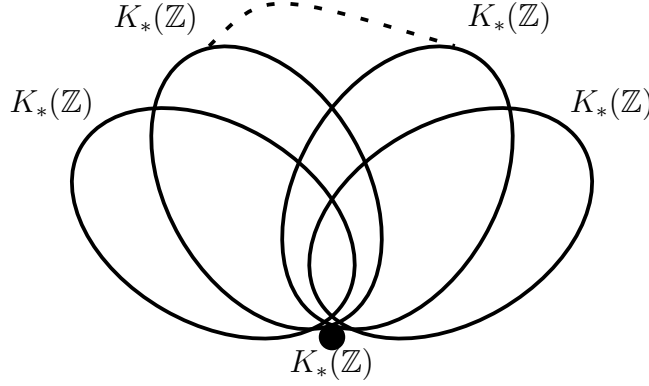
3.1. Finitely generated free groups. Let F_n be the free group on n generators, $n \in \mathbb{Z}$. Since F_n is torsion free $\underline{EG} = EG$, on the other hand we know that the Cayley graph of G is a model for EG , and BG with this model is a wedge of circles. Hence there is one 0-cell and n 1-cells. Moreover, $G_\sigma = 1$ for all cells, hence

$$E_{pq}^2 = H_p(\vee_n S^1; \{K_q\}) = H_p(\vee_n S^1; K_q(\mathbb{Z})).$$

This gives

$$H_p(\underline{BG}; K_q(\mathbb{Z})) = \begin{cases} K_q(\mathbb{Z}) & \text{for } p = 0, \\ \oplus_n (K_q(\mathbb{Z})) & \text{for } p = 1, \\ 0 & \text{for } p > 1 \text{ or } q \leq -1. \end{cases}$$

The graph associated to the free group on n generators, the labels are the coefficients of the corresponding cell, these have all trivial stabilizers.



Notice that all the differentials vanish, hence this spectral sequence collapses at this stage giving

$$\text{rank}(K_n(\mathbb{Z}[F_n])) = \text{rank}(K_n(\mathbb{Z})) + n \cdot \text{rank}(K_{n-1}(\mathbb{Z})).$$

Applying Theorem 3 to the trivial group we have that

$$\text{rank}(K_n(\mathbb{Z})) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{4} \text{ and } n > 1, \text{ or } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

it follows that

$$\text{rank}(K_n(\mathbb{Z}[F_n])) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{4} \text{ and } n > 1, \text{ or } n = 0, \\ n & \text{if } n \equiv 2 \pmod{4} \text{ and } n > 2, \text{ or } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

3.2. Free products of finite groups. Let G_1 and G_2 be finite groups, and let $G = G_1 * G_2$ be their free product. We can find a one-dimensional model for $\underline{E}G$ such that $\underline{B}G$ is a closed interval with trivial isotropy in the edge and isotropy at the vertices G_1 and G_2 ([Ser03]:

$$K_*(\mathbb{Z}[G_1]) \quad \xrightarrow{K_*(\mathbb{Z})} \quad K_*(\mathbb{Z}[G_2])$$

Once again our spectral sequence collapses at the second page giving

$$\text{rank}(K_n(\mathbb{Z}[G])) = \text{rank}(K_n(\mathbb{Z}[G_1])) + \text{rank}(K_n(\mathbb{Z}[G_2])) - \text{rank}(K_{n-1}(\mathbb{Z}))$$

and

$$\text{rank}(K_n(\mathbb{Z}[G])) = \begin{cases} 1 & n = 0, \\ r_1 + r_2 - q_1 - q_2 & i = 1, \\ r_1 + r_2 - 1 & \text{if } n \equiv 1 \pmod{4}, n > 1, \\ c_1 + c_2 & \text{if } n \equiv 3 \pmod{4}, n > 1, \\ 0 & \text{otherwise.} \end{cases}$$

where r_i is the number of distinct real irreducible representations of G_i , $i = 1, 2$; c_i is the number of distinct real irreducible representations of complex type of G_i , $i = 1, 2$; and q_i is the number of distinct rational irreducible representations of G_i , $i = 1, 2$.

3.3. $PSL_2(\mathbb{Z})$. This is a particular case of the previous example since $PSL_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$. Using the notation from above, we set $G_1 = \mathbb{Z}[\mathbb{Z}_2]$ and $G_2 = \mathbb{Z}[\mathbb{Z}_3]$, hence $r_1 = 2$, $r_2 = 2$, $c_1 = 0$, $c_2 = 1$, $q_1 = 2$, $q_2 = 2$ and

$$\text{rank}(K_n(\mathbb{Z}[PSL_2(\mathbb{Z})])) = \begin{cases} 0 & \text{if } n = -1, \\ 1 & n = 0, \\ 0 & n = 1, \\ 3 & \text{if } n \equiv 1 \pmod{4}, n > 1, \\ 1 & \text{if } n \equiv 3 \pmod{4}, n > 1, \\ 0 & \text{otherwise .} \end{cases}$$

3.4. The fundamental group of a closed orientable aspherical surface. Let S_g be the orientable closed surface of genus $g > 1$. Since the universal covering of S_g is contractible we have that S_g is a model for $B\pi_1(S_g)$. Furthermore, S_g is a model for $\underline{B}\pi_1(S_g)$ as well. Moreover, these groups are hyperbolic as S_g is a compact surface that admits a metric with constant curvature -1 . Using the classical construction of S_g as the quotient of a $4g$ -agon we can give S_g a CW-structure consisting of one 0-cell, $2g$ 1-cells, and one 2-cell and they all have trivial isotropy. Hence the second term of the Atiyah-Hirzebruch spectral sequence has constant coefficients the K -theory of the integers:

$$H_p(S_g; K_q(\mathbb{Z})) = \begin{cases} K_q(\mathbb{Z}) & \text{for } p = 0, \\ \bigoplus_{2g} K_q(\mathbb{Z}) & \text{for } p = 1, \\ K_q(\mathbb{Z}) & \text{for } p = 2, \\ 0 & \text{for } p > 1 \text{ or } q < 0. \end{cases}$$

Once again all differentials are trivial and our spectral sequence collapses. This gives

$$\text{rank}(K_n(\mathbb{Z}[\pi_1(S_g)])) = \begin{cases} 1 & n = 0, 2 \text{ or } n \equiv 1, 3 \pmod{4}, n > 1, \\ 2g & \text{if } n = 1 \text{ or } n \equiv 2 \pmod{4}, n > 1, \\ 0 & \text{otherwise .} \end{cases}$$

3.5. Finitely generated virtually free groups. Let G be a finitely generated virtually free group, that is G has a finitely generated free subgroup of finite index. As finitely generated free groups are hyperbolic and a group with a δ -hyperbolic subgroup of finite index is hyperbolic, it follows that G is δ -hyperbolic as well. By Bass-Serre theory [Ser03] and by the work in [JPLP11], one can find a tree T on which G acts with finite stabilizers, that is T is a model for \underline{EG} . Let \mathbf{E} and \mathbf{V} be the set of edges and vertices of the graph of groups for G determined by T , this part is developed in [JPLP11][section 2.2]. In order to calculate the second page of our spectral sequence, observe that T has only cells of dimension 0 and 1 hence our page is

$$E_{p,q}^2 = \begin{cases} \operatorname{coker} \left(\bigoplus_{e \in \mathbf{E}} K_q(\mathbb{Z}[G_e]) \rightarrow \bigoplus_{v \in \mathbf{V}} K_q(\mathbb{Z}[G_v]) \right), & \text{for } p = 0, \\ \operatorname{ker} \left(\bigoplus_{e \in \mathbf{E}} K_q(\mathbb{Z}[G_e]) \rightarrow \bigoplus_{v \in \mathbf{V}} K_q(\mathbb{Z}[G_v]) \right), & \text{for } p = 1, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the differentials vanish and our spectral sequence collapses at this page hence

$$\begin{aligned} H_n^G(\underline{EG}; \{\mathcal{K}_q\}) = & \operatorname{coker} \left(\bigoplus_{e \in \mathbf{E}} K_n(\mathbb{Z}[G_e]) \rightarrow \bigoplus_{v \in \mathbf{V}} K_n(\mathbb{Z}[G_v]) \right) \\ & \oplus \operatorname{ker} \left(\bigoplus_{e \in \mathbf{E}} K_{n-1}(\mathbb{Z}[G_e]) \rightarrow \bigoplus_{v \in \mathbf{V}} K_{n-1}(\mathbb{Z}[G_v]) \right). \end{aligned}$$

In order to simplify the notation, let us define

$$\begin{aligned} E_n &= \bigoplus_{e \in \mathbf{E}} K_n(\mathbb{Z}[G_e]), \\ V_n &= \bigoplus_{v \in \mathbf{V}} K_n(\mathbb{Z}[G_v]), \\ \operatorname{Ker}_n &= \operatorname{ker}(E_n \rightarrow V_n) \text{ and} \\ \operatorname{Cok}_n &= \operatorname{coker}(E_n \rightarrow V_n). \end{aligned}$$

In this way we have that

$$H_n^G(\underline{EG}; \{\mathcal{K}_q\}) = \operatorname{Cok}_n \oplus \operatorname{Ker}_{n-1}.$$

These last groups depend on the graph structure of our tree with the stabilizers of the action, which are all finite.

3.6. $G = F_n \rtimes S_n$. This example is worked out in detail in [JPLP11][§3] for other purposes. Let $G = F_n \rtimes S_n$ with the symmetric group S_n on n letters, acting on the free group on n generators by permuting the generators. The graph of groups is a single loop with vertex group S_{n-1} and edge group S_n . In this case the morphisms

$$E_i \rightarrow V_i$$

are all zero, it follows that

$$H_i^G(\underline{EG}; \{\mathcal{K}_q\}) = K_i(\mathbb{Z}[S_n]) \oplus K_{i-1}(\mathbb{Z}[S_{n-1}]).$$

It is well known that the conjugation class of an element $x \in S_n$ is determined by its cyclic decomposition, since x and x^{-1} have the same cyclic decomposition we have that they belong to the same conjugacy class. Hence the number of real conjugacy classes of S_n is equal to $p(n)$, the number of partitions of n , and the number of real conjugacy classes of complex type is zero. Finally if two elements on S_n determine the same cyclic subgroup then they are conjugate, this implies that the number of conjugacy classes of cyclic subgroups of S_n is equal to $p(n)$.

$$\text{rank}(K_i(\mathbb{Z}S_n)) = \begin{cases} p(n) & i \equiv 1 \pmod{4} \ i > 1, \\ 1 & i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\text{rank}(K_i(\mathbb{Z}G)) = \begin{cases} p(n) & i \equiv 1 \pmod{4} \ i > 1, \\ p(n-1) & i \equiv 2 \pmod{4} \ i > 1, \\ 1 & i = 0, \ 1, \\ 0 & \text{otherwise.} \end{cases}$$

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